Subsequences and rational recursions for solutions of polynomial difference equations

Computer Algebra in Scientific Computing – CASC 2026

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C-finite and holonomic sequences

- Fibonacci sequence.

$$F_{n+2} - F_{n+1} - F_n = 0,\ F_0 = 0, F_1 = 1.\ (F_n)_n := (0, 1, 1, 2, 3, 5, 8, 13, \ldots).$$

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- Fibonacci subsequence: $(f_n)_n \coloneqq (F_{dn})_n$, $d \in \mathbb{N}, d \geq 1$.

$$f_{n+2} - (\varphi^d + \phi^d) f_{n+1} + \varphi^d \phi^d f_n = 0,\ \varphi = \frac{1+\sqrt{5}}{2},\ \phi = \frac{1-\sqrt{5}}{2}.$$

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- Catalan sequence.

$$(n+2)C_{n+1} - 2(2n+1)C_n = 0,\ C_0 = 1.\ (C_n)_n \coloneqq (1, 1, 2, 5, 14, 42, 132, \ldots)$$

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- Rational recursion for the Catalan sequence [T. and Worrell, 2024].

$$C_{n+2} = \frac{2C_{n+1}(8C_n + C_{n+1})}{10C_n - C_{n+1}},\ C_0 = 1,\ C_1 = 1.\ (C_n)_n \coloneqq (1, 1, 2, 5, 14, 42, 132, \ldots)$$

Sequences defined by polynomial difference equations

- Let $\mathbb{K} \supset \mathbb{C}$, $r \in \mathbb{N}$, and $p \in \mathbb{K}[X_0, \ldots, X_r]$.

$$(s_n)_n : p(s_n, s_{n+1}, \ldots, s_{n+r}) = 0, \;\; s_0, s_1, \ldots, s_N \text{ given}, \; N \geq r.$$

Are Bernoulli numbers D-algebraic?

$$\begin{aligned} B_0 = 1,\ B_{2n} &= \frac{(-1)^{n+1} 2\,(2n)!}{(2\pi)^{2n}} \zeta(2n),\ n \geq 1, \\ B_1 = -\frac{1}{2},\ B_{2n+1} &= 0,\ n \geq 1, \end{aligned} \tag{1}$$

$$(B_n)_{n\in\mathbb{N}} = \left(1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}, \dots\right). \tag{2}$$

It holds that

$$5\,B_{n+3}\,B_n - 6\,B_{n+2}\,B_{n+1} + B_{n+1}\,B_n = 0, \text{ for all } n \in \mathbb{N}. \tag{3}$$

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For sufficiently large n,

$$q_{s,n}(x) := p(s_n, s_{n+1}, \ldots, s_{n+r-1}, x) \in \mathbb{K}[x] \setminus \{0\} \quad \text{and} \quad q_{s,n}(s_{n+r}) = 0.$$

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- What can we say about $(s_{dn})_n$ for $d \in \mathbb{N}, d \geq 1$?

C^2-finite sequences

Definition ([Jiménez-Pastor, Nuspl, and Pillwein, 2023])

The sequence $(s_n)_n$ is C^2-finite if there exist C-finite sequences $(c_{0,n}), (c_{1,n})_n, \ldots, (c_{r,n})_n$, with $c_{r,n} \neq 0$ for large n, such that

$$c_{0,n}s_n + c_{1,n}s_{n+1} + \cdots + c_{r,n}s_{n+r} = 0.$$

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  $\longrightarrow$ roots of characteristic polynomials or subsequences.
- If $(s_n)_n$ is C^2-finite then it is D-algebraic.
  $\longrightarrow$ Can be deduced from [Jiménez-Pastor, Pillwein, and Singer, 2020].
- Claim: if $(s_n)_n$ is C^2-finite then it is D-algebraic with linear $q_{s,n}$.

Outline

1. D-algebraic sequences
2. D-algebraic subsequences
3. Rational recursions

Language: difference algebra (R. Cohn, 1965)

- Let $(\mathbb{K}, \sigma)$ be a difference field, where $\mathbb{K}$ is an algebraically closed field of characteristic zero and σ an endomorphism in $\mathbb{K}$.
- $\mathcal{D}_\sigma(\mathbb{K}, x) := \mathbb{K}\left[\sigma^j(x) \mid j \in \mathbb{N}\right] = \mathbb{K}[\sigma^\infty(x)]$: the ring of difference polynomials in one difference indeterminate x. Note that $\sigma^{j+1}(x) = \sigma(\sigma^j(x)), j \in \mathbb{N}$.
- A difference ideal is an ideal in $\mathcal{D}_\sigma(\mathbb{K}, x)$ that is closed under application of σ. We denote by $< p >$, the minimal difference ideal defined by $p, \sigma(p), \sigma^2(p), \ldots$

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- The order of a difference polynomial $p \in \mathcal{D}_\sigma(\mathbb{K}, x)$, denoted $\mathrm{ord}(p)$, is the maximum $j \in \mathbb{N}$ such that $\sigma^j(x)$ appears in p.
- The degree of p denoted $\deg(p)$ is the total degree of p as a polynomial in the algebraic polynomial ring $\mathbb{K}[x, \sigma(x), \ldots, \sigma^{\mathrm{ord}(p)}(x)]$. We say that p is rationalizing when the degree of p with respect to $\sigma^{\mathrm{ord}(p)}(x)$ is 1.
- For any $p \in \mathcal{D}_\sigma(\mathbb{K}, x)$, $p = 0$ is called an algebraic difference equation.

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Proposition

Let $p \in \mathcal{D}_\sigma(\mathbb{K}, x) \setminus \mathbb{K}$. We have $\mathrm{ord}(\mathcal{I}_p) < \mathrm{ord}(p)$ and $\deg(\mathcal{I}_p) < \deg(p)$.

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Definition (Zero and "D-algebraic")

Let $p \in \mathcal{D}_\sigma(\mathbb{K}, x)$ and $(s_n)_n \in (\mathbb{K}^{\mathbb{N}}, \sigma' \equiv \text{shift})$. We say that $(s_n)_n$ is a zero of p if, under the unique homomorphism of difference rings $\mathcal{D}_\sigma(\mathbb{K}, x) \longrightarrow (\mathbb{K}^{\mathbb{N}}, \sigma')$ given by the extension that sends x to $(s_n)_n$, p is sent to 0. We may write "$p((s_n)_n) = 0$".
Moreover, $(s_n)_n$ is *D-algebraic* if $\mathcal{I}_p((s_n)_n) \neq 0$

A general definition of D-algebraicity for sequences is under investigation... [T., 2025a]

Closure properties: an example of division

- We can identify $\mathcal{D}_\sigma(\mathbb{K}, x)$ with $\mathbb{D}_s := \mathbb{K}[\sigma^j(s(n))|j \in \mathbb{N}]$, σ being clear from the context. For $p \in \mathbb{D}_s$, $(s(n))_n$ is viewed as a *D-algebraic (sequence) variable*.

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- Let $p = \sigma^2(x) - \sigma(x)\,x$, $q = \sigma(x) - x^2 - x$.
- Consider $(u_n)_n : u_0 = 1$, $u_1 = k \in \mathbb{K}$, $p((u_n)_n) = 0$. $u_n = k^{F_n}$, with the nth Fibonacci F_n (see A000301).
- And $(v_n)_n : v_0 = 1, v_1 = 2$, $q((v_n)_n) = 0$. $(v_n)_n$ is known to denote the number of ordered trees having nodes of outdegree $0, 1, 2$ and such that all leaves are at level n (see A007018).
- In $\mathbb{D}_u$, $p = u(n+2) - u(n+1)\,u(n)$ and $q = v(n+1) - v(n)^2 - v(n) \in \mathbb{D}_v$.

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- In $\mathbb{D}_u$, $p = u(n+2) - u(n+1)\,u(n)$ and $q = v(n+1) - v(n)^2 - v(n) \in \mathbb{D}_v$.
- We consider $\mathcal{D}_\sigma(\mathbb{K}, \{w_1, w_2, w_3, z\})$, with the encoding:

$$w_1 \to u(n),\ \ w_2 \to v(n),\ \ w_3 \to v(n+1),\ \ z \to \frac{w_1}{w_3}.$$

Closure properties: an example of division

We build the dynamical system:

$$\begin{cases} \sigma(w_1) = w_2 \\ \sigma(w_2) = w_1\, w_2 \\ \sigma(w_3) = w_3^2 + w_3 \\ z = \frac{w_1}{w_3} \end{cases}. \tag{4}$$

Find the difference polynomial of minimal degree among those of minimal order in

$$< \sigma(w_1) - w_2,\, \sigma(w_2) - w_1\, w_2,\, \sigma(w_3) - w_3^2 - w_3,\, w_3\, z - w_1 >: \{\sigma^j(w_3)\}^{\leq M} \cap \mathcal{D}_\sigma(\mathbb{K}, z).$$

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We get

$$\begin{aligned} &\Big\langle \sigma(z)^4\sigma^3(z)^2z^4 - z^3\sigma(z)^4\sigma^2(z)^2\sigma^3(z) - z^3\sigma(z)^3\sigma^2(z)\sigma^3(z)^2 - z^2\sigma(z)^3\sigma^2(z)^3\sigma^3(z) \\ &+ z\sigma(z)^3\sigma^2(z)^5 - z^2\sigma(z)^2\sigma^2(z)^2\sigma^3(z)^2 + 2z\sigma(z)^2\sigma^2(z)^4\sigma^3(z) + \sigma(z)^2\sigma^2(z)^6 \\ &+ z\sigma(z)\sigma^2(z)^3\sigma^3(z)^2 + 2\sigma(z)\sigma^2(z)^5\sigma^3(z) + \sigma^2(z)^4\sigma^3(z)^2 \Big\rangle. \end{aligned} \tag{5}$$

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Hence $(u_n/v_n)_n$ satisfies

$$\begin{aligned} & s(n+1)^4 s(n+3)^2 s(n)^4 - s(n+1)^4 s(n+3) s(n)^3 s(n+2)^2 + s(n+3)^2 s(n+2)^4 \\ & - s(n+1)^3 s(n+3)^2 s(n)^3 s(n+2) + s(n+1)^3 s(n) s(n+2)^5 + s(n+1)^2 s(n+2)^6 \\ & + 2s(n+1)^2 s(n+3) s(n) s(n+2)^4 - s(n+1)^2 s(n+3)^2 s(n)^2 s(n+2)^2 \\ & + s(n+1) s(n+3)^2 s(n) s(n+2)^3 + 2s(n+1) s(n+3) s(n+2)^5 \\ & - s(n+1)^3 s(n+3) s(n)^2 s(n+2)^3 = 0. \end{aligned} \tag{7}$$

Subsequences

- Let $(s_n)_n$ be D-algebraic of order r defined with some $p \in \mathcal{D}_\sigma(\mathbb{K}, x)$.
- A subsequence of a sequence $(s_n)_n$ is a sequence $(t_n)_n$ such that $t(n) = s_{\varphi(n)}$, where φ is a strictly increasing embedding of $\mathbb{N}$.

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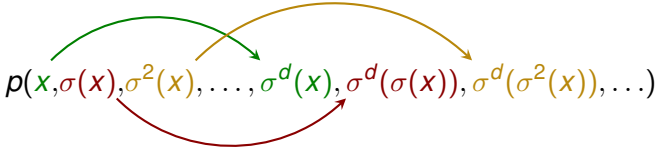
- We want to find a difference polynomial q so that $q((t_n)_n) = 0$.
- We need an endomorphism $\tilde{\sigma}$ that sends $s_{\varphi(n)}$ to $s_{\varphi(n+1)}$.
- For arithmetic progressions, $\varphi(n) = dn$, $d \in \mathbb{N}, d \geq 1$.

Subsequences at arithmetic progressions

Assume that x effectively occurs in p and $d \leq r$.

$$p(x, \sigma(x), \ldots, \sigma^r(x))$$



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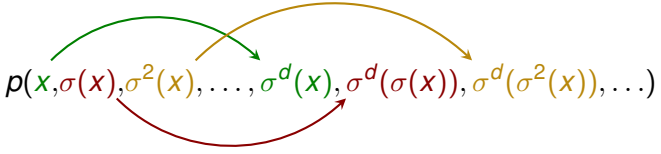

- $r = d\,r_1 + r_0$.
- $\tilde{\sigma} = \sigma^d$.
- $w_j \longrightarrow s_{dn+j}$, $j < d$.

$$p(w_0, w_1, \ldots, w_{d-1}, \tilde{\sigma}(w_1), \tilde{\sigma}(w_2), \ldots, \tilde{\sigma}^{r_1}(w_{r_0}))$$

Subsequences: case of Catalan numbers with $d = 3 > r = 2$

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$$s(n+2) = \frac{2s(n+1)\,(8s(n) + s(n+1))}{10s(n) - s(n+1)},\ s(0) = C_0,\ s(1) = C_1.$$

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We use $\sigma(p)$ and $\sigma^2(p)$ and build the system:

$$\begin{cases} \tilde{\sigma}(w_0) = \frac{2(16w_0 - w_1)(8w_0 + w_1)w_1}{(7w_0 - w_1)(10w_0 - w_1)} \\ \tilde{\sigma}(w_1) = \frac{4(16w_0 - w_1)(8w_0 + w_1)w_1(8w_0 - w_1)}{(6w_0 - w_1)(7w_0 - w_1)(10w_0 - w_1)} \\ z = w_0 \end{cases}.$$

Subsequences: $(C_{3n})_n$

$(C_{3n})_n$ satisfies the following second-order algebraic difference equation of degree 6.

$$
\begin{aligned}
&343597383680s(n)^3s(n+1)^3 - 69004689408s(n)^3s(n+1)^2s(n+2) + 4274823168s(n)^3s(n+1)s(n+2)^2 \\
&- 83243160s(n)^3s(n+2)^3 - 1258291200s(n)^2s(n+1)^4 + 266514432s(n)^2s(n+1)^3s(n+2) \\
&- 26883000s(n)^2s(n+1)^2s(n+2)^2 + 1043658s(n)^2s(n+1)s(n+2)^3 - 122880s(n)s(n+1)^5 \\
&- 101544s(n)s(n+1)^4s(n+2) + 65067s(n)s(n+1)^3s(n+2)^2 - 4113s(n)s(n+1)^2s(n+2)^3 \\
&+ 1400s(n+1)^6 - 30s(n+1)^5s(n+2) - 75s(n+1)^4s(n+2)^2 + 5s(n+1)^3s(n+2)^3 = 0. \quad (8)
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Conjecture ([T., 2025b])

$(C_{dn})_n$, $d \in \mathbb{N}\setminus\{0\}$, is D-algebraic of order 2 and degree $2d$.

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Conjecture ([T., 2025b])

$(C_{dn})_n$, $d \in \mathbb{N}\setminus\{0\}$, is D-algebraic of order 2 and degree $2d$.

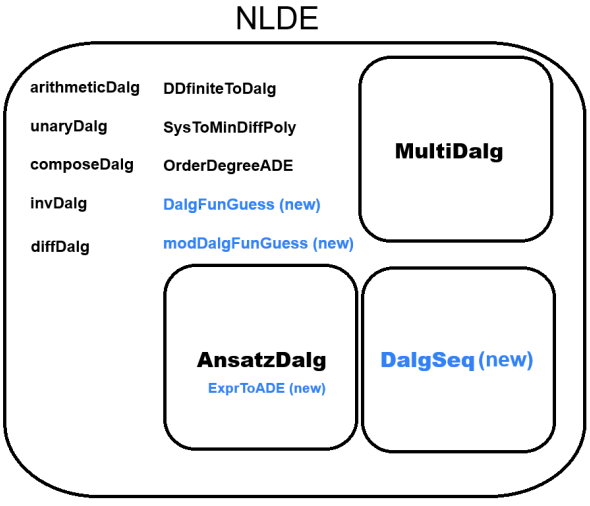
Theorem ([T., 2025a])

Let d be a positive integer. If $(s_n)_n$ is D-algebraic of order r, then $(s_{dn+j})_n$ is D-algebraic of order at most r, $0 \leq j < d$.

3. Rational recursions

Implementation

https://github.com/T3gu1a/D-algebraic-functions



Gröbner bases

Method I: elimination with Gröbner bases

>

The C^2-finite equation.

The inputs for the linear algebra approach and the Groebner bases approach differ by the presence of the index variable in the C-finite coefficient.

> $C2feq1 := c[1](n)\cdot s(n+1) + c[0](n)\cdot s(n) = 0$

$$C2feq1 := c_1(n)\, s(n+1) + c_0(n)\, s(n) = 0$$

The C-finite equations for the coefficients

> $Cfeq1 := c[1](n+2) = c[1](n+1) + c[1](n)$

$$Cfeq1 := c_1(n+2) = c_1(n+1) + c_1(n)$$

> $Cfeq2 := c[0](n+2) = c[0](n+1) + 3\cdot c[0](n)$

$$Cfeq2 := c_0(n+2) = c_0(n+1) + 3\, c_0(n)$$

Gröbner bases

```
> CPUTime( DalgSeq:-CCfiniteToDalg(C2feq2, s(n), [Cfeq1, Cfeq2], [c[1](n), c[0](n)]))
```

$$100.781,\ s(n+4)\,s(n+3)\,s(n+1)\,s(n)+s(n+4)\,s(n+2)^2\,s(n)-4\,s(n+3)^2\,s(n+2)\,s(n)$$
$$-6\,s(n+4)\,s(n+2)\,s(n+1)^2+3\,s(n+3)^2\,s(n+1)^2+9\,s(n+3)\,s(n+2)^2\,s(n+1)=0$$

```
> solve((1.4)[2], s(n+4))
```

$$-\frac{s(n+3)\left(3\,s(n+3)\,s(n+1)^2-4\,s(n+3)\,s(n)\,s(n+2)+9\,s(n+1)\,s(n+2)^2\right)}{s(n+3)\,s(n+1)\,s(n)+s(n)\,s(n+2)^2-6\,s(n+1)^2\,s(n+2)}$$

```
>
```

Linear algebra

Method II: linear algebra

>

The C^2-finite equation.

> $C2feq2 := c[1]\cdot s(n+1) + c[0]\cdot s(n) = 0$

$$C2feq2 := c_1\, s(n+1) + c_0\, s(n) = 0$$

> $CPUTime(DalgSeq{:}\text{-}CCfiniteToSimpleRatrec(C2feq1, s(n), [Cfeq1, Cfeq2], [c[1](n), c[0](n)]))$

$$0.187,\ s(n+4) = -\frac{s(n+3)\left(3\,s(n+3)\,s(n+1)^2 - 4\,s(n+3)\,s(n)\,s(n+2) + 9\,s(n+1)\,s(n+2)^2\right)}{s(n+3)\,s(n+1)\,s(n) + s(n)\,s(n+2)^2 - 6\,s(n+1)^2\,s(n+2)}$$

>

Thank You very much!

Some References

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