D-algebraicity for holonomic sequences

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Some References

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On rational recursive sequences.
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‘Ratrec’ and ‘simple ratrec’

Symbolic equivalence between ‘ratrec’ and ‘simple ratrec’ [Clemente et al., 2023]

Let $R_i, i = 1, \ldots, k$, be rational functions over a field $\mathbb{K}$ of characteristic zero, and $j \in \{1, \ldots, k\}$. Then there exists a rational function R such that the sequence $(s(n))_{n\in\mathbb{N}}$ defined with the discrete-time dynamical system

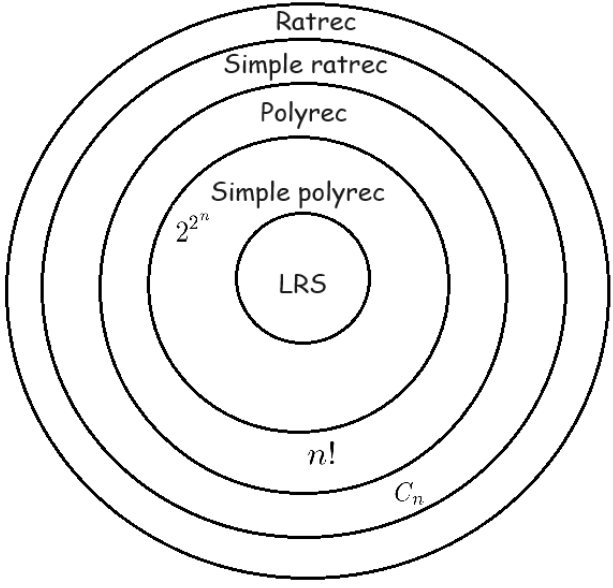
$$\begin{cases} a_i(0) = x_i,\ i = 1, \ldots, k, \\ a_i(n+1) = R_i(a_1(n), \ldots, a_k(n)),\ i = 1, \ldots, k, \quad , \ n \in \mathbb{N} \coloneqq \{0, 1, \ldots\}, \\ s(n) = a_j \end{cases} \tag{1}$$

satisfies the rational recursion

$$s(n+l) = R(s(n), \ldots, s(n+l-1)). \tag{2}$$

The initial values of (1) are distinct polynomial variables (symbols).

Question



Ratrec = Simple ratrec ?

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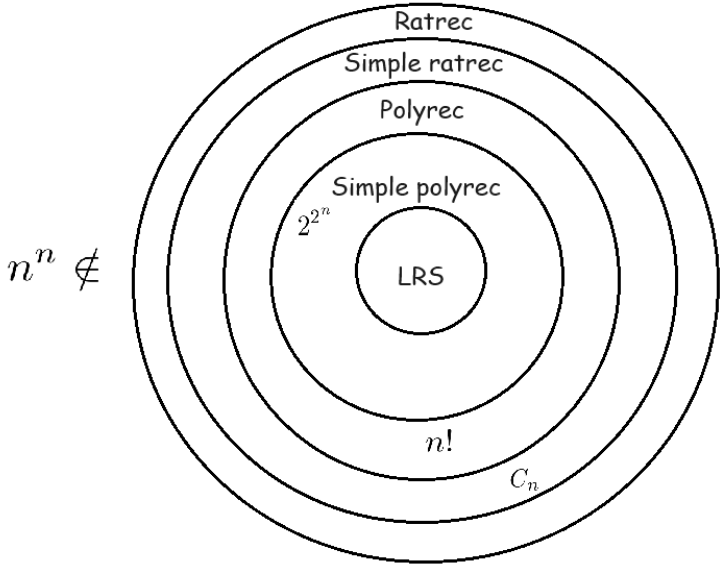

Ratrec = Simple ratrec ?

P-recursive sequence

Definition (Holonomic sequence)

A sequence $(s(n))_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$, is said to be holonomic if there exist polynomials $P_0, P_1, \ldots, P_m \in \mathbb{K}[X]$, $P_l \neq 0$, such that

$$P_0(n)\, s(n) + P_1(n)\, s(n+1) + \cdots + P_l(n)\, s(n+l) = 0. \tag{3}$$

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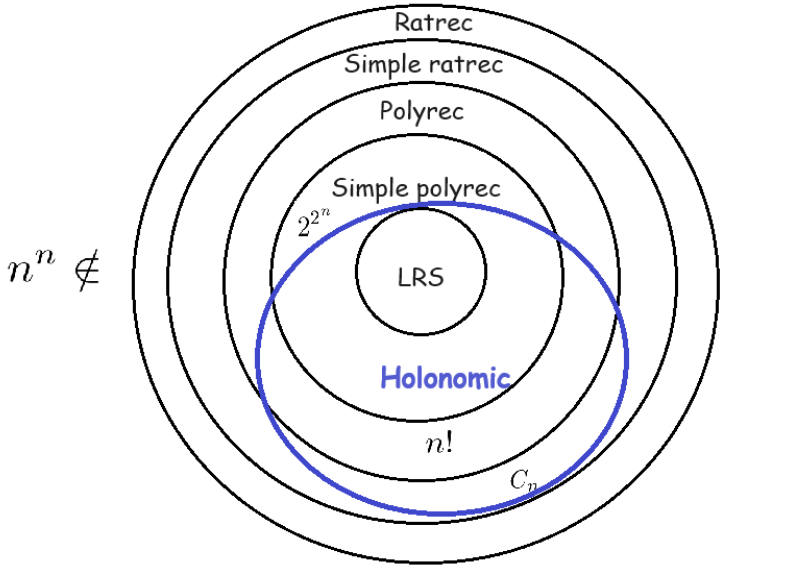
How to draw the class of P-recursive sequences in the previous diagram?

Main result in this talk



Ratrec = Simple ratrec ?

Outline

1. Introduction

2. D-algebraic sequences

3. Rational recursions for holonomic sequences

Preliminaries

- Let $(\mathbb{K}, \sigma)$ be a difference field, and σ an endomorphism in $\mathbb{K}$.
- $\mathcal{D}_\sigma(\mathbb{K}, x) \coloneqq \mathbb{K}\left[\sigma^j(x) \mid j \in \mathbb{N}\right] = \mathbb{K}[\sigma^\infty(x)]$: the ring of difference polynomials in one difference indeterminate x. Note that $\sigma^{j+1}(x) = \sigma(\sigma^j(x)), j \in \mathbb{N}$.
- A difference ideal is an ideal in $\mathcal{D}_\sigma(\mathbb{K}, x)$ that is closed under application of σ. We denote by $< p >$, the minimal difference ideal defined by $p, \sigma(p), \sigma^2(p), \ldots$

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- A difference ideal is an ideal in $\mathcal{D}_\sigma(\mathbb{K}, x)$ that is closed under application of σ. We denote by $< p >$, the minimal difference ideal defined by $p, \sigma(p), \sigma^2(p), \ldots$
- The order of a difference polynomial $p \in \mathcal{D}_\sigma(\mathbb{K}, x)$, denoted $\mathrm{ord}(p)$, is the maximum $j \in \mathbb{N}$ such that $\sigma^j(x)$ appears in p.
- The degree of p denoted $\deg(p)$ is the total degree of p as a polynomial in the algebraic polynomial ring $\mathbb{K}[x, \sigma(x), \ldots, \sigma^{\mathrm{ord}(p)}(x)]$. We say that p is rationalizing when the degree of p with respect to $\sigma^{\mathrm{ord}(p)}(x)$ is 1.
- For any $p \in \mathcal{D}_\sigma(\mathbb{K}, x)$, $p = 0$ is called an algebraic difference equation.

Initial and generic zeros

Definition

Let $p \in \mathcal{D}_\sigma(\mathbb{K}, x)$. The initial $\mathcal{I}_p$ of p is the leading coefficient of p viewed as a univariate polynomial in $\sigma^{\mathrm{ord}(p)}(x)$.

Proposition

Let $p \in \mathcal{D}_\sigma(\mathbb{K}, x) \setminus \mathbb{K}$. We have $\mathrm{ord}(\mathcal{I}_p) < \mathrm{ord}(p)$ and $\deg(\mathcal{I}_p) < \deg(p)$.

Definition

Let $p \in \mathcal{D}_\sigma(\mathbb{K}, x)$. A sequence $(s(n))_{n\in\mathbb{N}}$ is a generic zero of p if it is a zero of all P in the difference ideal $< p >$ and not a zero of any $P \notin < p >$.

Proposition

If a sequence $(s(n))_{n\in\mathbb{N}}$ is a generic zero of an algebraically irreducible difference polynomial $p \in \mathcal{D}_\sigma(\mathbb{K}, x)$, then $\mathcal{I}_p((s(n))_{n\in\mathbb{N}}) \neq 0$.

D-algebraic zeros

Definition (Definition 2.3 in [T., 2024-2025])

Let $p \in \mathcal{D}_\sigma(\mathbb{K}, x)$, and $p = p_0 p_1 \cdots p_\ell$, its decomposition into irreducible components. Let $\mathbb{N}_{p_j}(s(n)) \coloneqq \{n \in \mathbb{N},\ p_j(s(n)) = 0\}$. A sequence $(s(n))_{n\in\mathbb{N}}$ is difference-algebraic (or D-algebraic) over $\mathbb{K}$ with defining difference polynomial p, if:

1. For all $j \in \{0, \ldots, \ell\}$, $|\mathbb{N}_{p_j}(s(n))| = \infty$;
2. For all $j \in \{0, \ldots, \ell\}$, $|\{n \in \mathbb{N}_{p_j}(s(n)),\ \mathcal{I}_{p_j}(s(n)) = 0\}| < \infty$;
3. For all $n \in \mathbb{N}$, there exists a unique $j \in \{0, \ldots, \ell\}$ such that $p_j(s(n)) = 0$.

D-algebraic sequences

Example

Let $p \coloneqq (2x\sigma(x) + x + \sigma(x))\,(2\sigma(x)x - x - \sigma(x))$. Using p and the initial term $s(0) \coloneqq 1$, we define the recursion:

$$s(n+1) \coloneqq \begin{cases} \frac{s(n)}{2\cdot s(n)-1}, \text{ if } n \text{ is odd,} \\ -\frac{s(n)}{2\cdot s(n)+1}, \text{ if } n \text{ is even.} \end{cases} \tag{4}$$

The sequence $(s(n))_{n\in\mathbb{N}}$ is D-algebraic and we have $s(n) = \frac{(-1)^n}{2n+1}$, for all $n \in \mathbb{N}$.

D-algebraic sequences

Example

Let $p := (2x\sigma(x) + x + \sigma(x))\,(2\sigma(x)x - x - \sigma(x))$. Using p and the initial term $s(0) := 1$, we define the recursion:

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The sequence $(s(n))_{n\in\mathbb{N}}$ is D-algebraic and we have $s(n) = \frac{(-1)^n}{2n+1}$, for all $n \in \mathbb{N}$.
Using the algorithm from [T. and Worrell, 2024] one shows that $(s(n))_{n\in\mathbb{N}}$ can also be defined with the rationalizing difference polynomial

$$q := (2\,x + \sigma(x))\,\sigma^2(x) + x\sigma(x).$$

When does an irreducible q exist for a given reducible p and a sequence it defines?

3. Rational recursions for holonomic sequences

Settings

- We replace the field $\mathbb{K}$ with $\mathbb{K}(n)$ and work on the specialized ring of difference polynomials $\mathbb{D}_s(n) \coloneqq \mathbb{K}(n)[\sigma^{\infty}(s(n))]$.

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Definition

The holonomic degree of a difference polynomial $p \in \mathbb{D}_s(n)$ of degree 1 is the degree of p viewed as a univariate polynomial in n.

Series coefficients of $\arctan(z)$

Let $\arctan(z) = \sum_{n=0}^{\infty} a(n) z^n$.
Then $(a(n))_{n\in\mathbb{N}}$ satisfies the holonomic equation

$$n s(n) + (n+2)\, s(n+2) = 0, \tag{5}$$

with $a(0) = 0, a(1) = 1$.

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with $a(0) = 0, a(1) = 1$.

Using the algorithm from [T. and Worrell, 2024] to convert (5) into a rationalizing ADE (or rational recursion) yields

$$s(n+3) = -\frac{s(n+1)\,(-s(n+2) + s(n))}{s(n+2) + 3s(n)}. \tag{6}$$

However, since the zero terms cannot be computed with this recursion, $(a(n))_{n\in\mathbb{N}}$ cannot be defined with (6).

Series coefficients of $\arctan(z)$

Treating the terms of odd and even indices separately, we shows that $(a(n))_{n\in\mathbb{N}}$ is a D-algebraic zero of

$$\sigma^2(x)\left((\sigma^2(x)+2x)\sigma^4(x)+\sigma^2(x)x+\sigma(x)\right)\in\mathcal{D}_\sigma(\mathbb{K},x) \tag{7}$$

All terms of the sequence can be computed as follows

$$a(n)=\begin{cases} a(n-2) & \text{if } n \text{ is even} \\ -\frac{a(n-2)a(n-4)}{a(n-2)+2a(n-4)} & \text{otherwise.}\end{cases} \tag{8}$$

The “almost all” concept [T., 2024-2025]

Definition (Almost all solutions of a holonomic equation)

Let $p \in \mathbb{D}_s(n)$ be a holonomic difference polynomial of order l_1, and $q \in \mathbb{D}_s$ of order l_2, $l_2 \geq l_1$. Let $\tilde{p} = \sigma^{N+1}(p)$, where N is the maximum nonnegative integer root of the polynomial coefficients in p. Let $(\epsilon_p(n))_{n>N}$ be the solution of $\tilde{p}$ with symbolic initial terms

$$\epsilon_p(j+N) = x_j \in \{x_1, \ldots, x_{l_1}\},$$

where the x's are viewed as distinct polynomial variables. We say that almost all zeros of p are D-algebraic zeros of q if $(\epsilon_p(n))_{n>N}$ is a generic zero of q.

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For the series coefficients of $\arctan(z)$, $N = 0$, and the corresponding $(\epsilon(n))_{n>0}$ is the sequence

$$\left(x_1, x_2, -\frac{x_1}{3}, -\frac{x_2}{2}, \frac{x_1}{5}, \frac{x_2}{3}, -\frac{x_1}{7}, -\frac{x_2}{4}, \frac{x_1}{9}, \frac{x_2}{5}, -\frac{x_1}{11}, -\frac{x_2}{6}, \frac{x_1}{13}, \ldots\right). \tag{9}$$

Almost all and almost every

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- However, the corresponding symbolic solution $(\epsilon(n))_{n>N}$ is not a generic zero of q, as it is a zero of $\mathcal{I}_q = \sigma(p)$.

Definition (Almost every holonomic sequence)

We say that almost every holonomic sequence is a D-algebraic zero of a family of difference polynomials $\mathcal{F}$, if for any holonomic difference polynomial p, almost all its zeros are D-algebraic zeros of some $q \in \mathcal{F}$.

Main result of this talk

Theorem (Theorem 4.5 in [T., 2024-2025])

Almost every holonomic sequence of order l and degree d is a D-algebraic zero of a rationalizing difference polynomial of order at most $l + d$ in $\mathbb{D}_s$.

Steps of the proof

1. Let $p \in \mathbb{D}_s(n)$ be the minimal holonomic difference polynomial that vanish at $(s(n))_{n\in\mathbb{N}}$ with order l and degree d.

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2. We look at the equation $p = 0$ and its shifts as polynomial equations in n such that

$$\sigma^j(p) = 0 \iff \sum_{k=1}^{d} \gamma_{j,k}\, n^k = -\gamma_{j,0},\ j = 0, \ldots, d-1,\ \gamma_{j,k} \in \mathbb{K}[s(n+j), \ldots, s(n+j+l)].$$

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   This is a linear system of d equations in $(n, n^2, \ldots, n^d)^T$.
3. Show that the matrix of the system has a generic full rank due to the shifts involved in the γ's. This entails proving that the determinant of the system is a nonzero polynomial in $\mathbb{K}[s(n), \ldots, s(n+l+d-1)]$.

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5. The coefficient of $s(n+d+l)$ is nonzero because it yields a linear equation with constant coefficient that is linearly independent from the system of the n^k's.
6. Substituting the corresponding $(\epsilon_p(j))_{j>N}$ into $\mathcal{I}_q$ evaluates the corresponding polynomial to j.

Automatic computation in Maple

```
> with(NLDE:-DalgSeq, HoloToSimpleRatrec)
```

$$[HoloToSimpleRatrec]$$

```
> HRE := HyperTypeSeq:-HolonomicRE( binomial(2·n, n)/(n+1), a(n) )
```

$$HRE := (n+2)\,a(n+1) - (4n+2)\,a(n) = 0$$

```
> HoloToSimpleRatrec(HRE, a(n))
```

$$a(n+2) = \frac{2\,a(n+1)\,(8\,a(n) + a(n+1))}{10\,a(n) - a(n+1)}$$

```
> HRE := FPS:-FindRE(sin(x) + cos(x), x, a(n))
```

$$HRE := (n+2)\,(n+1)\,a(n+2) + a(n) = 0$$

```
> HoloToSimpleRatrec(HRE, a(n))
```

$$a(n+4) = -\frac{a(n+2)^2\,a(n+3)}{a(n)\,a(n+3) - 2\,a(n+1)\,a(n+2) + 2\,a(n+2)\,a(n+3)}$$

```
> HoloToSimpleRatrec(a(n+1) = (n+1)^2·a(n), a(n))
```

$$a(n+3) = \frac{a(n+2)\,\left(2\,a(n)\,a(n+1) + 2\,a(n)\,a(n+2) - a(n+1)^2\right)}{a(n)\,a(n+1)}$$

```
>
```

Thank You!